Photograph from National Geographic, Sept 2008

An interior-point stochastic approximation method and an L1-regularized delta rule

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A motivating example

- **Goal:** $p(spam \mid email\text{-}features,\ \theta)$.
  (Cormack and Lynam, 2005)
- **Approach:** find model θ that maximizes likelihood of training data.
- **But:** training data is observed over time (on-line learning).
- **Moreover:** we need to penalize complex models θ.

What is stochastic approximation, briefly

- **Original Problem:** (Spall, 2003; Kushner and Yin, 2003; Bottou, 1998)
  1. Minimize $f(x)$, or find $F(x) = \nabla f(x) = 0$.
  2. We only observe **noisy**, **unbiased** $g_k \approx F(x_k)$.
- **Robbins & Monro algorithm:**
  1. $x_{k+1} = x_k - a_k g_k$
  2. $\{a_k\}$ is a sequence of decreasing step sizes.
- **Problem in this talk:**

$$\begin{aligned} &\text{minimize} && f(x) \\ &\text{subject to} && c(x) \leq 0. \end{aligned}$$

Motivating example
(continued)

- **nonsmooth**, **unconstrained** objective:

$$\text{minimize} \quad -\log p(\mathit{spam} \mid \mathit{email\text{-}features},\ \theta) + \lambda \| \theta \|_1$$

- change θ to x to obtain **smooth**, **constrained** objective:

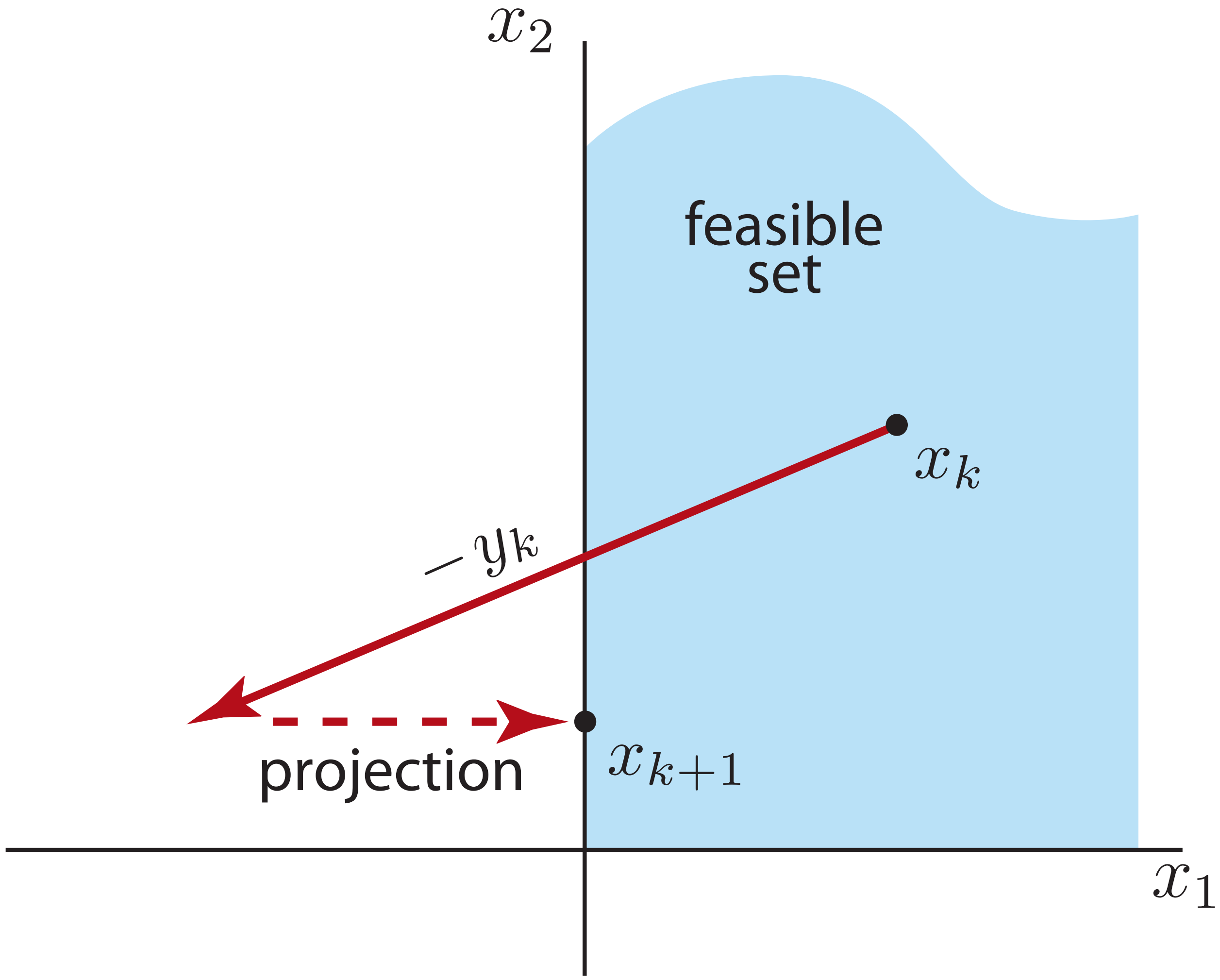
$$\begin{aligned} &\text{minimize} && -\log p(\mathit{spam} \mid \mathit{email\text{-}features},\ x) + \lambda \textstyle\sum_i x_i \\ &\text{subject to} && x \geq 0. \end{aligned}$$

$$\| u \|_1 \equiv \textstyle\sum_i |u_i|$$

One possible approach

Projected gradient
(Bertsekas, 1999; Poljak, 1978)

- ✓ Has convergence guarantees.
- ✗ Not always efficient to compute projection.
- ✗ Big steps may be biased ⇒ slow progress.



The interior-point approach

Projected gradient

- ✓ Has convergence guarantees.
- ✗ Only feasible for simple constraints.
- ✗ Large steps may be biased.

Primal-dual Interior-point method

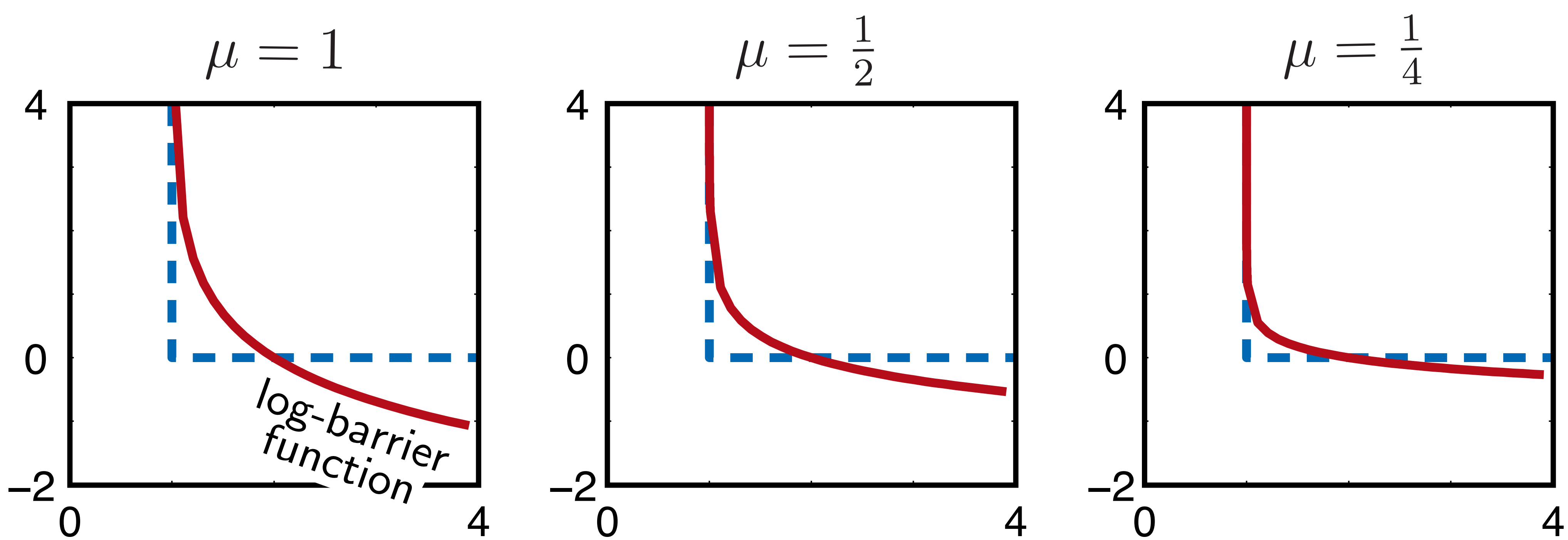
- ✓ Also has convergence guarantees.
- ✓ Works for many types of constraints.
- ✓ Steps are never biased.

The interior-point approach

- **Our problem:**
  minimize $f(x)$
  subject to $c(x) \leq 0$.

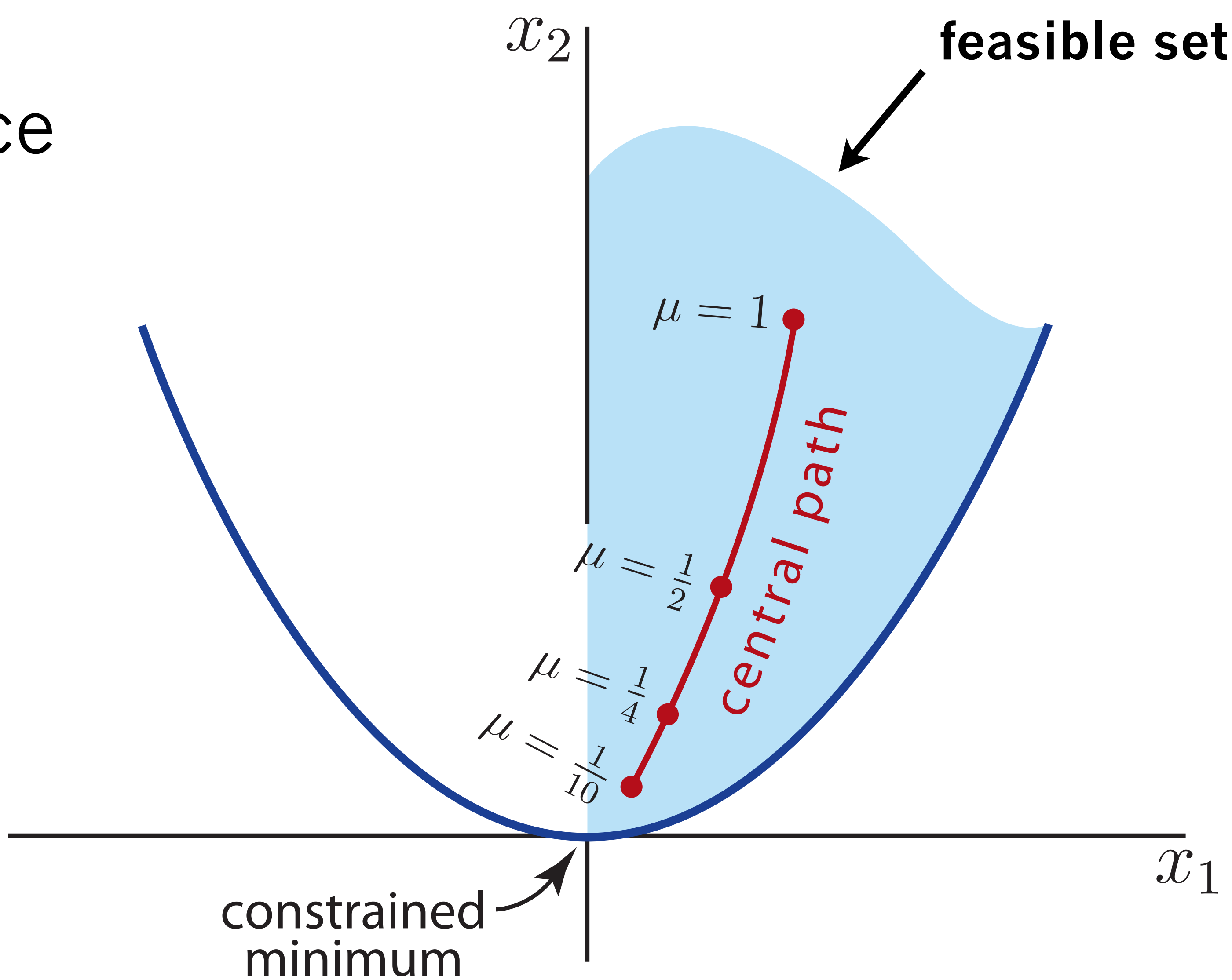
- **The barrier function:**
  (Fiacco and McCormick, 1968)

$$f_\mu(x) \equiv f(x) - \mu \sum_{i=1}^{m} \log(-c_i(x))$$



The interior-point approach

- **Primal interior-point method:** solve sequence $F_\mu(x) = \nabla f_\mu(x) = 0$ for decreasing $\mu > 0$.
- **But:** we cannot assess convergence to each subproblem $F_\mu(x) = 0$!



Adapted from Fiacco and McCormick (1968).

The "primal-dual" approach

(M. H. Wright, 1992; S. J. Wright, 1996; *many others*...)

- Recall problem: minimize $f(x)$ subject to $c(x) \leq 0$.

"dual" variables

- Introduce Lagrange multiplier-like variables z.
- Use Robbins-Monro to solve **moving target**:

$$F_\mu(x,y) \equiv \begin{bmatrix} \nabla f(x) + \nabla c(x)^T z \\ c(x) \bullet z + \mu \end{bmatrix} = \begin{bmatrix} \text{gradient of Lagrangian} \\ \text{complementarity} \end{bmatrix} = 0.$$

"Perturbed" KKT conditions

- Take steps

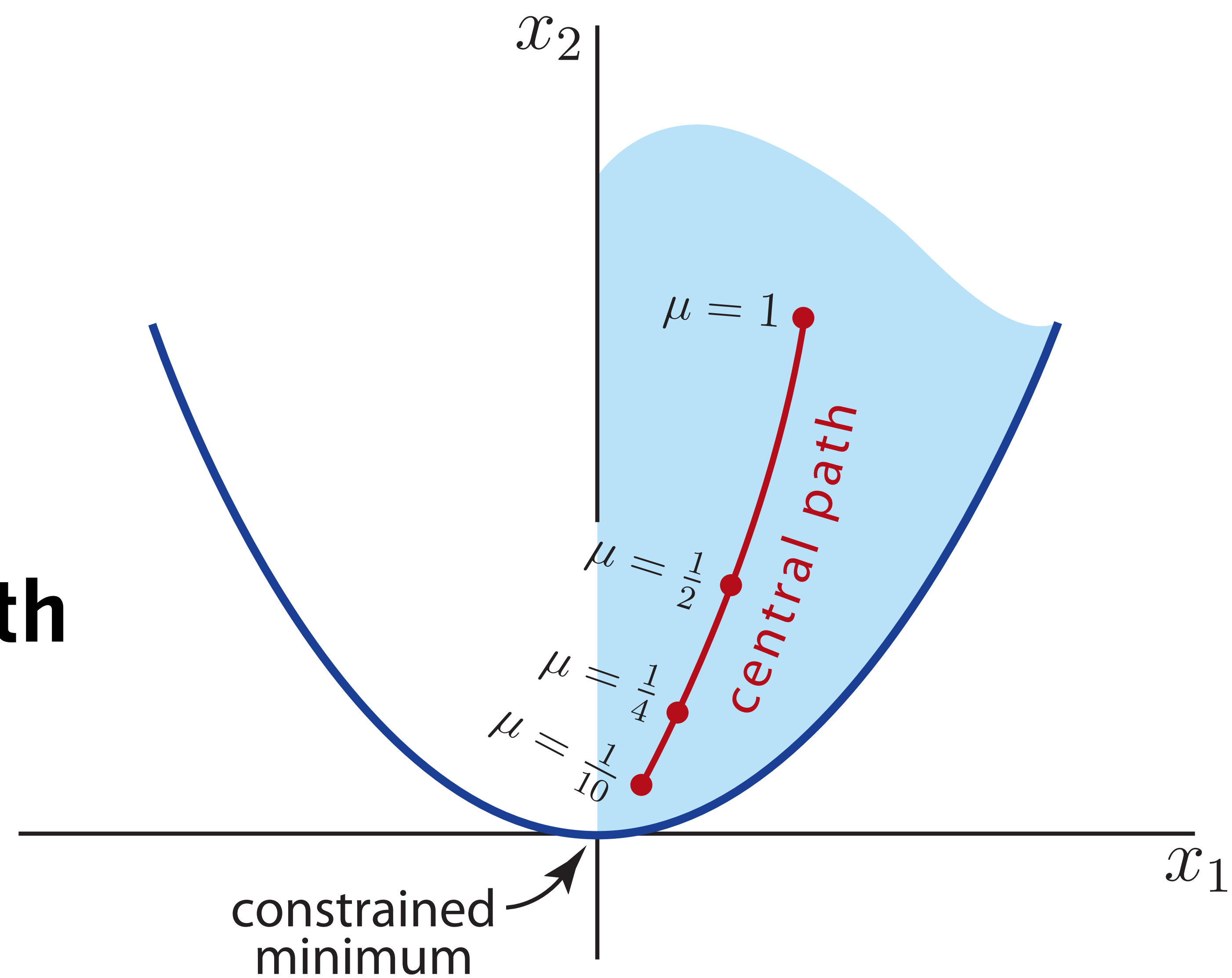
$$x_{k+1} = x_k + \hat{a}_k \boxed{\begin{matrix} \Delta x_k \\ \Delta z_k \end{matrix}}$$
$$z_{k+1} = z_k + \hat{a}_k$$

Replace with "noisy" estimate

Primal-dual search direction

Why does this work?

1. Central path ⇒ numerically stable.
2. Primal-dual search direction keeps us on central path, **even with noisy gradients**.



A small experiment

- **Problem:** linear regression + L1 penalty (Lasso)

$$\begin{array}{ll} \text{minimize} & \|Ax - b\|^2 + \lambda \sum_i x_i \\ \text{subject to} & x \geq 0. \end{array}$$

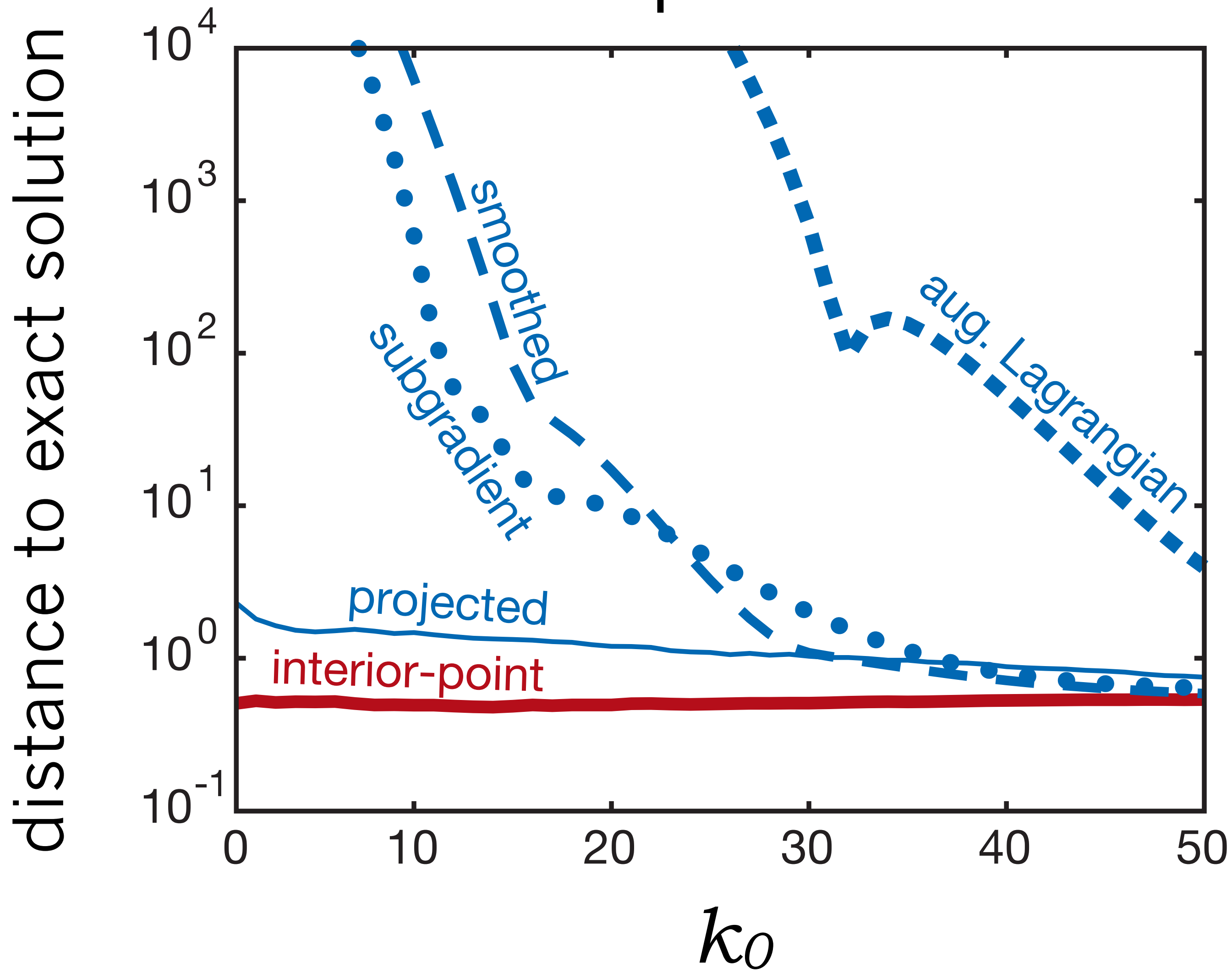
- **Question:** how well does on-line estimate recover exact solution?
- Synthetic data.
- Repeat for $k = 1$ to 100, step sizes $a_k = 1/(k_0 + k)$.

A small experiment

- Compared these methods:
  - Projected gradient
  - Primal-dual interior-point
  - Sub-gradient (Shalev-Shwartz et al, 2007; Hazan, 2007)
  - Smoothed approximation
  - Augmented Lagrangian (Wang and Spall, 2003)

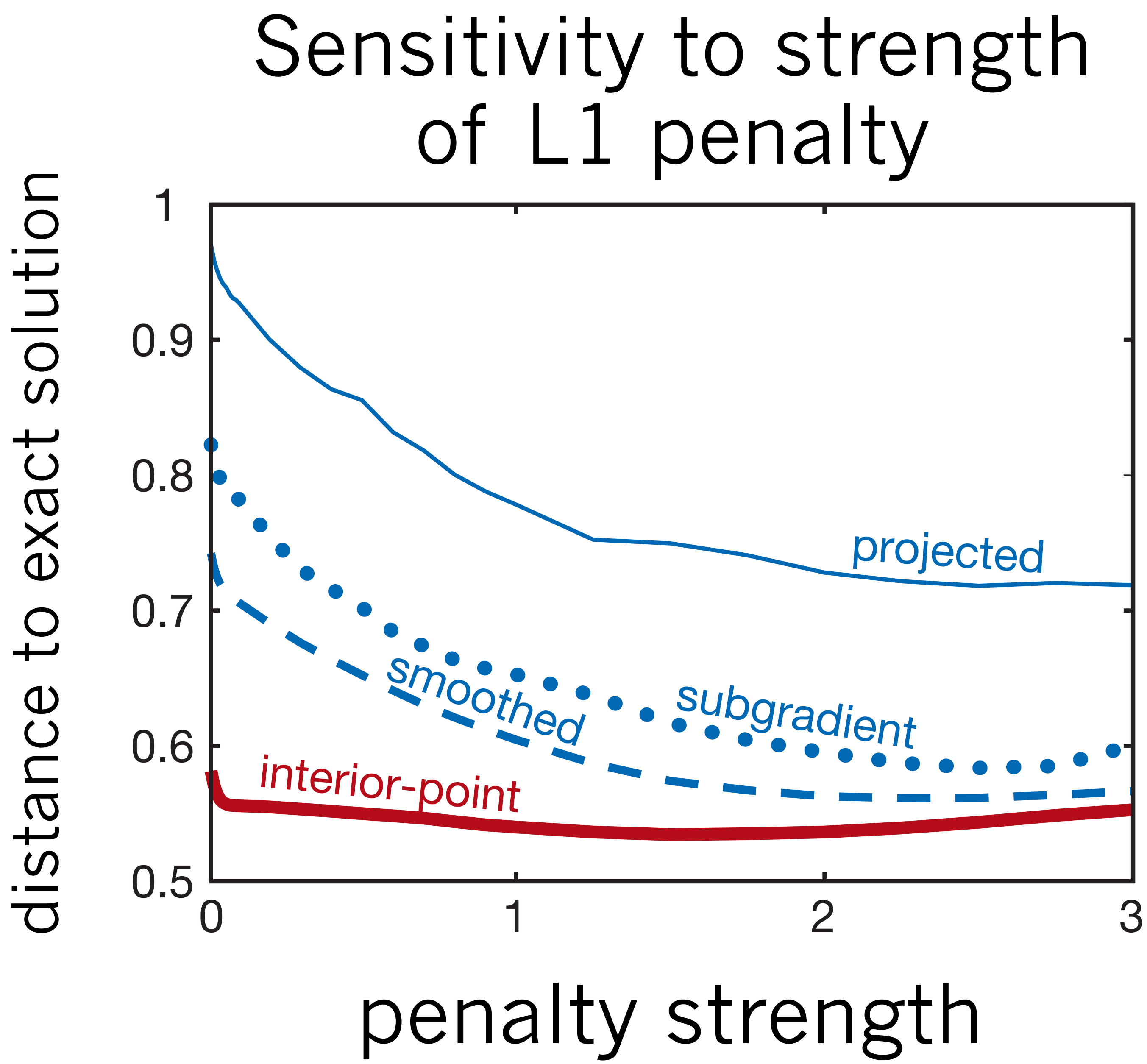
Some empirical evidence

Sensitivity to step size sequence

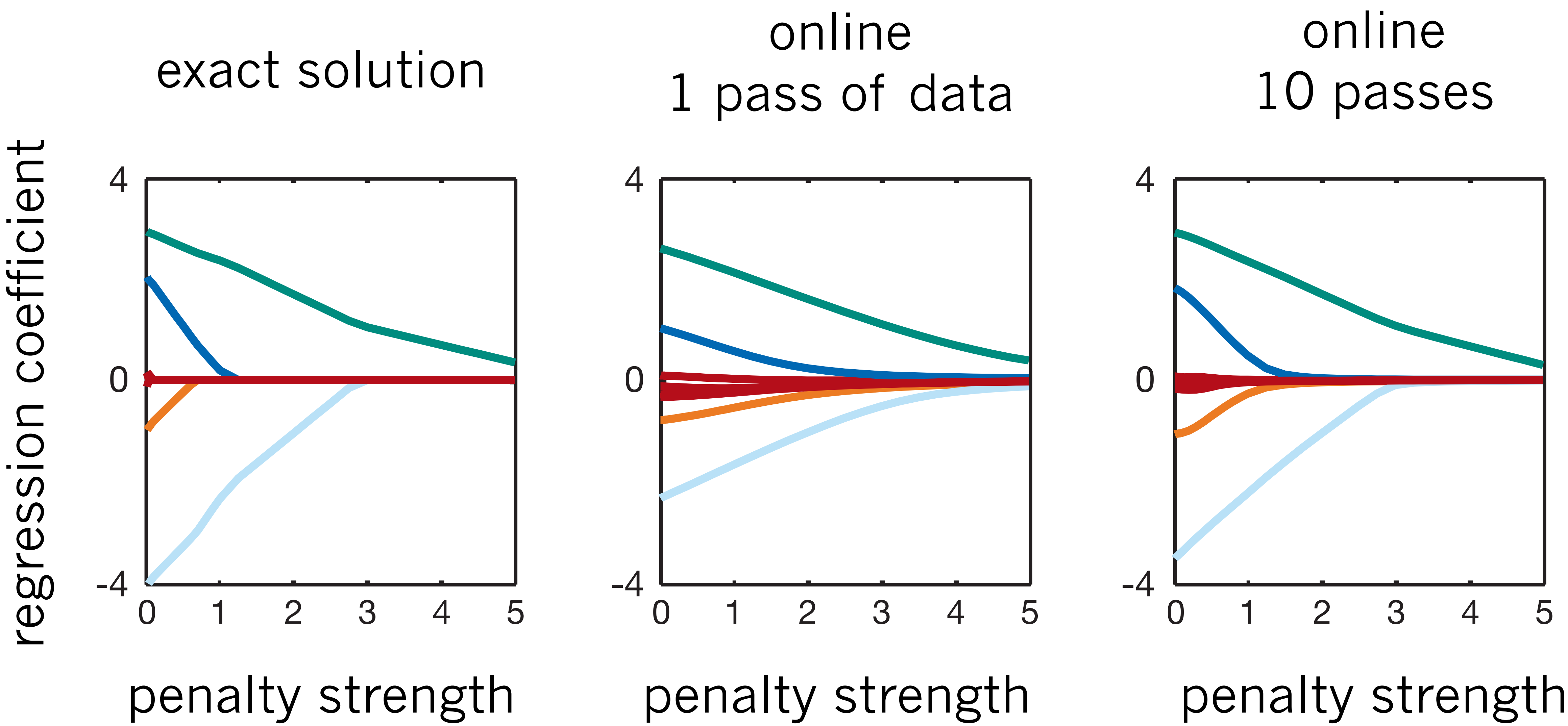


(step sizes are $a_k = 1/(k_0 + k)$.)

Some empirical evidence



Shrinkage effect



In summary

- Robbins-Munro solved $F(x) = 0$ with updates

$$\boxed{x_{k+1} = x_k - a_k g_k.}$$

- We solve sequence $F_\mu(x,z) = 0$ with updates

$$\boxed{\begin{aligned} x_{k+1} &= x_k + \hat{a}_k \Delta x_k \\ z_{k+1} &= z_k + \hat{a}_k \Delta z_k \end{aligned}}$$

  where $(\Delta x, \Delta z)$ is the solution to the "perturbed" KKT conditions.

Thank you!